Corrigé CC2 - ondes du 25-04-24

Problème I 8pts

3pts ① $f_{n+1\to n} = -K(x_n - x_{n+1})$, 1pt $f_{n-1\to n} = -K(x_n - x_{n-1})$ 1pt

$2^{ème}$ loi de Newton : 1pt

$$\boxed{m\ddot{x}_n = -K(x_n - x_{n+1}) - K(x_n - x_{n-1})}$$

$$\Rightarrow \quad \ddot{x}_n + \omega^2(2x_n - x_{n-1} - x_{n+1}) = 0, \quad \omega^2 = K/m$$

3pts ② 1pt

$$x_{n+1}(t) = F((n+1)a_0, t) \simeq F(na_0, t) + \left.\frac{\partial F}{\partial x}\right)_{x=na_0} a_0 + \frac{1}{2}\left.\frac{\partial^2 F}{\partial x^2}\right)_{x=na_0} a_0^2$$

1pt

$$x_{n-1}(t) = F((n-1)a_0, t) \simeq F(na_0, t) - \left.\frac{\partial F}{\partial x}\right)_{na_0} a_0 + \frac{1}{2}\left.\frac{\partial^2 F}{\partial x^2}\right)_{x=na_0} a_0^2$$

d'où $\ddot{x}_n + \omega^2(2x_n - x_{n-1} - x_{n+1}) = 0 \quad \Rightarrow$

$$\frac{\partial^2 F(na_0, t)}{\partial t^2} + \omega^2\left(\cancel{2F(na_0,t)} - \cancel{2F(na_0,t)} - \left.\frac{\partial^2 F}{\partial x^2}\right)_{x=na_0} a_0^2\right) = 0$$

1pt

$$\boxed{\frac{\partial^2 F(x,t)}{\partial t^2} - \omega^2 a_0^2 \frac{\partial^2 F(x,t)}{\partial x^2} = 0}$$

← équation d'onde

$$\frac{\partial^2 F(x,t)}{\partial x^2} - \frac{1}{\omega^2 a_0^2}\frac{\partial^2 F(x,t)}{\partial t^2} = 0$$

2pts ③ 1pt

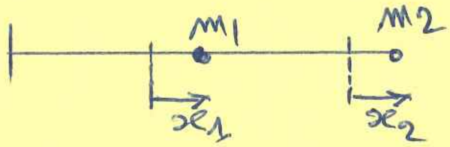
$$\frac{1}{a_0^2\omega^2} = \frac{1}{c^2} \Rightarrow c = a_0\omega, \quad a_0 = \frac{1}{k}$$ où k est le nombre d'onde

1pt

c est la vitesse de phase

Problème II 12pts

2pts ①



$\vec{F_1} = -K x_1 \vec{U_x} - K(x_1 - x_2)\vec{U_x}$ 1pt

$\vec{F_2} = -K(x_2 - x_1)\vec{U_x}$ 1pt

2pts ② $\vec{F} = m\vec{a} \Rightarrow \begin{cases} \ddot{x}_1 = \omega_1^2 (x_2 - 2x_1), & \omega_1^2 = \frac{K}{m_1} \quad \text{1pt} \\ \ddot{x}_2 = \omega_2^2 (x_1 - x_2), & \omega_2^2 = \frac{K}{m_2} \quad \text{1pt} \end{cases}$

$$\Rightarrow \begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{pmatrix} + \underbrace{\begin{pmatrix} 2\omega_1^2 & -\omega_1^2 \\ -\omega_2^2 & \omega_2^2 \end{pmatrix}}_{= W} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$$ 1pt

2pts ③ les valeurs propres λ^2 : $\left| \begin{pmatrix} 2\omega^2 - \lambda^2 & -\omega^2 \\ -\omega^2 & \omega^2 - \lambda^2 \end{pmatrix} \right| = 0$ 1pt

$\Rightarrow (2\omega^2 - \lambda^2)(\omega^2 - \lambda^2) - \omega^4 = 0$

$\lambda^4 - 3\omega^2\lambda^2 + \omega^4 = 0$ *

les pulsations propres $\boxed{\omega_\pm = \lambda_\pm = \omega \left(\frac{3 \pm \sqrt{5}}{2}\right)^{1/2}}$ 1pt

calcul: * $\rightarrow \lambda_\pm^2 = \dfrac{3\omega^2 \pm \sqrt{9\omega^4 - 4\omega^4}}{2} = \omega^2 \left(\dfrac{3 \pm \sqrt{5}}{2}\right)$

2pts ④ vect. propres $\begin{pmatrix} a_\pm \\ b_\pm \end{pmatrix}$ tel que $\begin{pmatrix} 2\omega^2 - \lambda_\pm^2 & \omega^2 \\ -\omega^2 & \omega^2 - \lambda_\pm^2 \end{pmatrix} \begin{pmatrix} a_\pm \\ b_\pm \end{pmatrix} = 0$

$\Rightarrow \begin{cases} (2\omega^2 - \lambda_\pm^2) a_\pm = \omega^2 b_\pm \\ \text{et } -\omega^2 a_\pm = (\lambda_\pm^2 - \omega^2) b_\pm \end{cases}$

$\Rightarrow \quad (\omega^2 - \lambda_\pm^2) a_\pm = \lambda_\pm^2 b_\pm$

$\boxed{\lambda_+^2}$ $(\omega^2 - \lambda_+^2) a_+ = \lambda_+^2 b_+ \Rightarrow \begin{cases} a_+ = \lambda_+^2 \\ \text{et } b_+ = \omega^2 - \lambda_+^2 \end{cases} \Rightarrow \vec{V_+} = \begin{pmatrix} \lambda_+^2 \\ \omega^2 - \lambda_+^2 \end{pmatrix}$ 1pt

$$\vec{V_-} = \begin{pmatrix} \lambda_-^2 \\ w^2 - \lambda_-^2 \end{pmatrix} \quad \text{avec } \lambda_-^2 = w_-^2$$

1pt

⑤ Modes propres :

2pts

$$x_\pm = e^{i w_\pm t} \vec{V_\pm} \quad \text{et leurs conjugués complexes}$$

1pt 1pt

2pts ⑥ $x(t) = a x_+ + b x_+^* + c x_- + d x_-^*$

$$\Rightarrow x(0) = (a+b) x_+\big|_{t=0} + (c+d) x_-\big|_{t=0} = 0$$

$$\Rightarrow a = -b \;,\; c = -d$$

$$\dot{x}(0) = (i\lambda_+ a - i\lambda_+ b) x_+\big|_{t=0} + (i\lambda_- c - i\lambda_- d) x_-$$

$$\dot{x}(0) = \begin{pmatrix} 0 \\ -v_0 \end{pmatrix}$$

$$\Rightarrow a = \frac{i v_0 (w^2 - \lambda_\pm^2)}{2\lambda_+ (\lambda_+^4 + (w^2 - \lambda_\pm^2)^2)}$$

$$c = \frac{i v_0 (w^2 - \lambda_-^2)}{2\lambda_- (\lambda_-^4 + (w^2 - \lambda_-^2)^2)}$$

$$\Rightarrow x(t) = 2ia (\sin(\lambda_+ t)) \vec{V_+} + 2ic (\sin(\lambda_- t)) \vec{V_-}$$

Erreur dans la correction :

2 pts ② $\vec{F} = m\vec{a} \Rightarrow \begin{cases} \ddot{x}_1 = \omega_1^2 (x_2 - 2x_1) \text{ 1pt } 0{,}5 , & \omega_1^2 = \dfrac{K}{m_1} \\ \ddot{x}_2 = \omega_2^2 (x_1 - x_2) \text{ 1pt } 0{,}5 , & \omega_2^2 = \dfrac{K}{m_2} \end{cases}$

$$\Rightarrow \begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{pmatrix} + \underbrace{\begin{pmatrix} 2\omega_1^2 & -\omega_1^2 \\ -\omega_2^2 & \omega_2^2 \end{pmatrix}}_{= W} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$$

1pt